

1 2-Category of Double Groupoids

This is a topic entry on the 2-category of double groupoids.

1.1 Introduction

Definition 1.1. Let us recall that if X is a [topological](#) space, then a *double groupoid* $\mathcal{D}$ is defined by the following [categorical diagram](#) of linked [groupoids](#) and sets:

$$\mathcal{D} := \begin{array}{ccc} & \xrightarrow{s^1} & \\ S & \xrightleftharpoons[t^1]{} & H \\ \uparrow s_2 & \begin{array}{c} \xrightarrow{t_2} \\ \xleftarrow{s} \end{array} & \uparrow s \\ V & \xrightleftharpoons[t]{} & M \end{array}, \quad (1.1)$$

where M is a set of points, H, V are two groupoids (called, respectively, “horizontal” and “vertical” groupoids) , and S is a set of [squares with two composition laws, \$\bullet\$ and \$\circ\$](#) (as first defined and represented in ref. [1] by Brown et al.) . A simplified notion of a [thin square](#) is that of “a continuous map from the unit square of the real plane into X which factors through a tree” ([1]).

1.2 Homotopy double groupoid and homotopy 2-groupoid

The [algebraic composition laws](#), $\bullet$ and $\circ$, employed above to define a [double groupoid](#) $\mathcal{D}$ allow one also to define $\mathcal{D}$ as a groupoid internal to the [category of groupoids](#). Thus, in the particular case of a Hausdorff space, X_H , a double groupoid called the [homotopy double groupoid](#) of X_H can be denoted as follows

$$\rho_2^\square(X_H) := \mathcal{D},$$

where $\square$ is in this case a [thin square](#). Thus, the construction of a homotopy double groupoid is based upon the geometric notion of thin square that extends the notion of thin relative homotopy as discussed in ref. [1]. One notes however a significant distinction between a homotopy [2-groupoid](#) and homotopy double groupoid construction; thus, the construction of the 2-cells of the homotopy double groupoid is based upon a suitable cubical approach to the notion of thin 3-cube, whereas the construction of the 2-cells of the homotopy 2-groupoid can be interpreted by means of a globular notion of thin 3-cube. “The homotopy double groupoid of a space, and the related homotopy 2-groupoid, are constructed directly from

the cubical singular complex and so (they) remain close to geometric intuition in an almost classical way" (viz. [1]).

1.3 Defintion of 2-Category of Double Groupoids

Definition 1.2. The [2-category](#), $\mathcal{G}^2$ – whose [objects](#) (or 2-cells) are the above [diagrams](#) $\mathcal{D}$ that define double groupoids, and whose 2-morphisms are [functors](#) $\mathbb{F}$ between double groupoid $\mathcal{D}$ diagrams– is called the *double groupoid [2-category](#)*, or the *2-category of double groupoids*.

Remark 1.1. $\mathcal{G}^2$ is a relatively simple example of a [category](#) of diagrams, or a 1-supercategory, §1.

References

- [1] R. Brown, K.A. Hardie, K.H. Kamps and T. Porter., [A homotopy double groupoid of a Hausdorff space](#) , *Theory and Applications of Categories* **10**,(2002): 71-93.
- [2] R. Brown and C.B. Spencer: Double groupoids and crossed modules, *Cahiers Top. Géom.Diff.*, **17** (1976), 343–362.
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- [4] K.A. Hardie, K.H. Kamps and R.W. Kieboom., A homotopy 2-groupoid of a Hausdorff *Applied Categorical Structures*, **8** (2000): 209-234.
- [5] Al-Agl, F.A., Brown, R. and R. Steiner: 2002, Multiple categories: the equivalence of a globular and cubical approach, *Adv. in Math.*, **170**: 711-118.